



**You have downloaded a document from
RE-BUS
repository of the University of Silesia in Katowice**

Title: Singular limit in a parabolic equation

Author: Tomasz Dłotko

Citation style: Dłotko Tomasz. (1993). Singular limit in a parabolic equation. "Demonstratio Mathematica" (Vol. 26, nr 2 (1993) s. 473-481), doi 10.1515/dema-1993-0218



Uznanie autorstwa - Użycie niekomercyjne - Bez utworów zależnych Polska - Licencja ta zezwala na rozpowszechnianie, przedstawianie i wykonywanie utworu jedynie w celach niekomercyjnych oraz pod warunkiem zachowania go w oryginalnej postaci (nie tworzenia utworów zależnych).



UNIwersytet ŚLĄSKI
W KATOWICACH



Biblioteka
Uniwersytetu Śląskiego



Ministerstwo Nauki
i Szkolnictwa Wyższego

Tomasz Dłotko

SINGULAR LIMIT IN A PARABOLIC EQUATION

1. Introduction

Our aim here is to study the singular limit, when $\varepsilon \rightarrow 0$, in a parabolic problem

$$(1) \quad \begin{cases} u_t = \varepsilon^2 \Delta u + f(t, u), & x \in \Omega \subset \mathbb{R}^n, t > 0, \\ u(0, x) = u_0(x) & \text{for } x \in \Omega, \\ Bu = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

with a bounded smooth domain Ω , $\varepsilon > 0$ and the boundary operator B of either the Dirichlet type ($B = \text{Id}$) or the Neumann type ($B = \frac{\partial}{\partial n}$, n is the normal vector to $\partial\Omega$). We want to estimate the difference between u and the solution $y = y(t, x)$ of the limit problem

$$(2) \quad \begin{cases} y_t = f(t, y), & t > 0, \\ y(0, x) = u_0(x), & x \in \Omega, \end{cases}$$

the central interest being estimates on bounded time interval $[0, T]$. The problem of long time behaviour (when $t \rightarrow \infty$) of a pair u, y has been considered previously, e.g. in [4, 7]. This paper has been prompted by recent studies on the so-called "bistable" reaction-diffusion equation and the phenomenon of transition layers (cf. [3, 8, 1, 12]). The simplest non-numerical way to study the preliminary phase of creation of the layer structure for the solution of (1) is to observe this phenomenon for the solution y of (2) and to give the ε -dependent estimates for the difference $(u - y)$ suitable for small ε .

2. Assumptions

The function f considered here is assumed to be of the class \mathbb{C}^1 with respect to t and \mathbb{C}^2 with respect to u on bounded subsets of $[0, \infty) \times \mathbb{R}$ and to have only three different zeros; $f(t, \pm 1) = f(t, 0) = 0$ with $f_u(t, \pm 1) < 0$ and $f_u(t, 0) \geq 0$. As a typical example of such f let us consider the function

$\frac{1}{2}(u - u^3)$. We denote

$$m = \sup f_u(t, u), \quad M = \sup f_{uu}(t, u),$$

where the supremums are taken over $[0, T] \times [-1, 1]$. Let $\partial\Omega \in \mathbb{C}^{2+\mu}$ (Hölder space) with some $\mu > 0$. We further assume that the initial function $u_0 \in \mathbb{C}^{2+\mu}(\bar{\Omega})$ satisfies the compatibility conditions:

$$u_0(x) = 0 = \Delta u_0(x) \quad \text{for } x \in \partial\Omega \text{ in the Dirichlet case,}$$

$$\frac{\partial u_0}{\partial n}(x) = 0 \quad \text{for } x \in \partial\Omega \text{ in the Neumann case.}$$

DEFINITION. A function $u \in L^\infty((0, T) \times \Omega) \cap L^2(0, T; V)$ ($V = H_0^1(\Omega)$ in the Dirichlet case, $V = H^1(\Omega)$ in the Neumann case) is said to be a weak solution of (1) in $D = (0, T) \times \Omega$, if the identity

$$(3) \quad \int_D [u\phi_t - \varepsilon^2 \nabla u \nabla \phi + f(t, u)\phi] dx dt = - \int_\Omega u_0 \phi(0, x) dx$$

holds for every $\phi \in \mathbb{C}^1(\bar{D})$ with $\phi(T, x) = 0$ and, in the Dirichlet case, $\phi(t, x) = 0$ for $x \in \partial\Omega$. Weak solutions of the problem (2) are defined in a similar way; we formally set $\varepsilon = 0$ in (3).

Remark 1. As long as we consider initial functions u_0 with values in the interval $[-1, 1]$, it is a familiar consequence of the Maximum Principle and the condition $f(t, \pm 1) = 0$ that the values of the corresponding solutions u also belong to $[-1, 1]$. Thus the form of f outside $[-1, 1]$ for such u_0 is invalid.

We will assume in the sequel that the values of u_0 belong to $[-1, 1]$. With all this assumptions there is a unique global solution u to (1), moreover with $u_t, \Delta u$ continuous in $[0, \infty) \times \bar{\Omega}$ (see [5, 11]) and u_{tx_i} belonging to $L^2(D)$ ([11], p. 513).

3. Introductory observation

Our first goal is to show that when ε tends to zero in (1) the corresponding weak solution u^ε tends to the weak solution y of (2). Let us first observe that for all ε , $|u^\varepsilon(t, x)| \leq 1$. Next, by the Schwarz inequality

$$\varepsilon \int_D \varepsilon \nabla u^\varepsilon \nabla \phi dx dt \leq \varepsilon \|\varepsilon \nabla u^\varepsilon\|_{L^2(D)} \|\nabla \phi\|_{L^2(D)},$$

hence the middle component in (3) will vanish when $\varepsilon \rightarrow 0$, provided we justify uniform in ε estimate of $\|\varepsilon \nabla u^\varepsilon\|_{L^2(D)}$.

LEMMA 1. *The quantity $\|\varepsilon \nabla u^\varepsilon\|_{L^2(D)}$ is bounded uniformly in ε for $\varepsilon \in [0, \varepsilon_0]$.*

Proof. To simplify notation we put $u^\varepsilon = u$ from now on. Multiplying the equation in (1) by Δu and integrating over $D_t = (0, t) \times \Omega$ we get

$$\iint u_t \Delta u \, dx \, ds = \varepsilon^2 \iint (\Delta u)^2 \, dx \, ds + \iint f(s, u) \Delta u \, dx \, ds$$
or integrating by parts, using the boundary condition and the estimate $f_u \leq m$,

$$(4) \quad \int_{\Omega} \sum_i u_{x_i}^2(t, x) \, dx - \int_{\Omega} \sum_i u_{0x_i}^2 \, dx + 2\varepsilon^2 \iint (\Delta u)^2 \, dx \, ds \leq 2m \iint \sum_i u_{x_i}^2(s, x) \, dx \, ds, \quad t \in (0, T].$$

As a first consequence of (4), using the Gronwall inequality, we get

$$(5) \quad \int_{\Omega} \sum_i u_{x_i}^2(t, x) \, dx \leq \int_{\Omega} \sum_i u_{0x_i}^2 \, dx e^{2mt}.$$

Next, from (4) and (5) we get

$$\begin{aligned} 2\varepsilon^2 \int_0^T \int_{\Omega} (\Delta u)^2 \, dx \, d\tau &\leq \int_{\Omega} \sum_i u_{0x_i}^2 \, dx \left(1 + 2m \frac{e^{2mT} - 1}{2m}\right) \\ &= \int_{\Omega} \sum_i u_{0x_i}^2 \, dx e^{2mT}. \end{aligned}$$

Basing on the property of the Newtonian potential [9], p. 235;

$$\left(\sum_{i,j} \|h_{x_i x_j}\|_{L^2(\Omega)}^2 \right)^{1/2} = \|\Delta h\|_{L^2(\Omega)},$$

valid for $h \in H_0^2(\Omega)$ and the estimate of intermediate derivatives (e.g. [9], p. 171) we find that the expression $(\|\Delta h\|_{L^2(\Omega)}^2 + \|h\|_{L^2(\Omega)}^2)^{1/2}$ defines an equivalent norm on $H_0^2(\Omega)$. For arbitrary $h \in H_0^2(\Omega)$ we thus have an estimate

$$(6) \quad \int_{\Omega} \sum_i h_{x_i}^2 \, dx \leq C \left(\int_{\Omega} (\Delta h)^2 \, dx + \int_{\Omega} h^2 \, dx \right)$$

($C = C(n, \Omega)$) and, continuing our previous calculations, we find that

$$\begin{aligned} (7) \quad \varepsilon^2 \int_0^T \int_{\Omega} \sum_i u_{x_i}^2 \, dx \, dt &\leq C\varepsilon^2 \int_0^T \int_{\Omega} (\Delta u)^2 \, dx \, dt + C\varepsilon^2 \int_0^T \int_{\Omega} u^2 \, dx \, dt \\ &\leq \frac{C}{2} \int_{\Omega} \sum_i u_{0x_i}^2 \, dx e^{2mT} + C\varepsilon_0^2 T |\Omega| \end{aligned}$$

($|\Omega|$ denotes the measure of Ω). The proof is completed.

Remark 2. For the case of the Neumann boundary condition, instead of (6) we will use an estimate ([4], p. 12)

$$(6') \quad \lambda \|\nabla h\|_{L^2(\Omega)}^2 \leq \|\Delta h\|_{L^2(\Omega)}^2$$

valid for every $h \in H^2(\Omega)$ with $\frac{\partial h}{\partial n} = 0$ on $\partial\Omega$ and the smallest positive eigenvalue λ of $-\Delta$ with homogeneous Neumann boundary condition.

At this point the usual procedure would be to use the compactness argument for the family $\{u^\varepsilon\}$ to be able to pass to the limit over a sequence $\varepsilon_n \rightarrow 0$. Instead we will give a proportional to ε estimate in $L^2(D)$ (and other L^p spaces) for the difference $u^\varepsilon - y$, to justify the limit passage generally when $\varepsilon \rightarrow 0$.

We shall start with some remarks concerning the properties and smoothness of the solution $y(t, x)$.

Boundary condition for y . It is easy to see that the (zero order) compatibility conditions required for u_0 are automatically preserved by equation (2) for y .

For the Dirichlet condition; $u(x_0) = 0$, $x_0 \in \partial\Omega$, with our assumption $f(t, 0) = 0$ we find that the solution of (2) satisfies $y(t, x_0) = 0$. For the Neumann case, when $x_0 \in \partial\Omega$, the normal derivative of y satisfies the equation

$$\frac{\partial}{\partial t} \left(\frac{\partial y}{\partial n}(t, x_0) \right) = f_y(t, y(t, x_0)) \frac{\partial y}{\partial n}(t, x_0)$$

or

$$(8) \quad \frac{\partial y}{\partial n}(t, x_0) = \frac{\partial y}{\partial n}(0, x_0) \exp \left(\int_0^t f_y(s, y(s, x_0)) ds \right) = 0$$

which is equal to zero because $\frac{\partial y}{\partial n}(0, x_0) = 0$.

The equation for y . It is clear that the dependence of y on x enters through the dependence of $y(0, x)$ on x . There is no other connection between $y(t, x_1)$ and $y(t, x_2)$ for $t > 0$.

We will now calculate the Laplacian of y . For f twice differentiable with respect to y we have

$$y_{tx_i} = f_y(t, y)y_{x_i}, \quad y_{tx_i x_i} = f_{yy}(t, y)y_{x_i}^2 + f_y(t, y)y_{x_i x_i},$$

so that

$$(9) \quad \frac{1}{2} \frac{\partial}{\partial t} \sum_i y_{x_i}^2 = f_y(t, y) \sum_i y_{x_i}^2,$$

$$(10) \quad \frac{\partial}{\partial t} \Delta y = f_{yy}(t, y) \sum_i y_{x_i}^2 + f_y(t, y) \Delta y.$$

Solving this system of equations we find an expression for $\sum_i y_{x_i}^2$:

$$(11) \quad \sum_i y_{x_i}^2(t, x) = \sum_i u_{0x_i}^2(x) \exp \left(2 \int_0^t f_y(\tau, y(\tau, x)) d\tau \right),$$

and, with the use of (11), also for Δy

$$(12) \quad \Delta y(t, x) = \left(\Delta u_0(x) + \int_0^t f_{yy}(\tau, y(\tau, x)) \sum_i u_{0x_i}^2(x) \right. \\ \left. \cdot \exp \left(\int_0^\tau f_y(s, y(s, x)) ds \right) d\tau \right) \exp \left(\int_0^t f_y(s, y(s, x)) ds \right).$$

Observe, that the solution y of (2) solves simultaneously the problem

$$(13) \quad \begin{cases} y_t = \varepsilon^2 \Delta y + f(t, y(t, x)) - \varepsilon^2 \Delta y, \\ y(0, x) = u_0(x), \quad x \in \Omega, \\ By = 0, \quad x \in \partial\Omega. \end{cases}$$

The "free term" $-\varepsilon^2 \Delta y$ is given by (12).

We are now ready to formulate the estimate of $w = u - y$ in L^p spaces.

LEMMA 2. For every $k \in \mathbb{N}$ the following estimate holds

$$(14) \quad \|w(t, \cdot)\|_{L^{2k}(\Omega)} \leq \varepsilon \|\nabla u_0\|_{L^{2k}(\Omega)} \left(\frac{2k-1}{2} t \right)^{1/2} e^{mt}.$$

Proof. The function w solves the problem

$$(15) \quad \begin{cases} w_t = \varepsilon^2 \Delta w + f_u(t, \tilde{u})w + \varepsilon^2 \Delta y, \\ w(0, x) = 0, \quad x \in \Omega, \\ Bw = 0, \quad x \in \partial\Omega. \end{cases}$$

Multiplying the equation in (15) by e^{-mt} ($f_u \leq m$) we get the equation for $W = we^{-mt}$

$$(16) \quad W_t = \varepsilon^2 \Delta W + (f_u - m)W + \varepsilon^2 \Delta(ye^{-mt}).$$

Multiplying (16) by W^{2k-1} and integrating over Ω , we find

$$\frac{1}{2k} \frac{d}{dt} \int_\Omega W^{2k} dx \leq -\varepsilon^2 \frac{2k-1}{k^2} \int_\Omega \sum_i (W^k)_{x_i}^2 dx \\ - \varepsilon^2 \frac{2k-1}{k} \int_\Omega \sum_i (ye^{-mt})_{x_i} W^{k-1} (W^k)_{x_i} dx =: -J_1 - J_2.$$

Now, from the Cauchy inequality

$$|J_2| \leq \varepsilon^2 \frac{2k-1}{k^2} \int_\Omega \sum_i (W^k)_{x_i}^2 dx + \frac{\varepsilon^2}{4} (2k-1) \int_\Omega \sum_i ((ye^{-mt})_{x_i})^2 W^{2k-2} dx,$$

and further, from the Hölder inequality ($p = \frac{k}{k-1}$, $q = k$) and (11)

$$|J_2| \leq J_1 + \frac{\varepsilon^2}{4}(2k-1) \left(\int_{\Omega} \left(\sum_i u_{0x_i}^2 \right)^k dx \right)^{1/k} \left(\int_{\Omega} W^{2k} dx \right)^{1-1/k}.$$

We achieve an estimate

$$\frac{d}{dt} \int_{\Omega} W^{2k}(t, x) dx \leq \frac{\varepsilon^2}{2} k(2k-1) \|\nabla u_0\|_{L^{2k}(\Omega)}^2 \left(\int_{\Omega} W^{2k}(t, x) dx \right)^{1-1/k}$$

or, when integrated

$$\|W(t, \cdot)\|_{L^{2k}(\Omega)}^2 \leq \frac{\varepsilon^2}{2} (2k-1) \|\nabla u_0\|_{L^{2k}(\Omega)}^2 t,$$

where we note that $W(0, x) = 0$ in Ω . The proof is completed.

For $k = 1$, integrating (14) over $[0, T]$, we obtain proportional to ε estimate of w in $L^2(D)$. This estimate together with Lemma 1 justify the limit passage in (3) with ε to 0. Weak solution of (1) tends to the weak solution of (2) (being also strong solution of (2)).

4. Main estimates

We are ready to give estimates of the difference $u - y$ more accurate than in Lemma 2.

THEOREM 1. *For the difference w the following estimate holds*

$$(17) \quad \|w(t, \cdot)\|_{L^\infty(\Omega)} \leq \varepsilon^2 \left\| |\Delta u_0| + \sum_i u_{0x_i}^2 M \frac{e^{mt} - 1}{m} \right\|_{L^\infty(\Omega)} t e^{mt},$$

with $f_u \leq m$ and $f_{uu} \leq M$ for $(t, x) \in [0, T] \times [-1, 1]$ (when $m = 0$ then the term $\frac{e^{mt} - 1}{m}$ in (17) should be replaced by t).

The proof follows directly from Proposition 1 if we apply it to W solving (16) and use the immediate estimate of $\varepsilon^2 \Delta y$ following from (12).

Application. Consider the standard nonlinearity $f(z) = \frac{1}{2}(z - z^3)$. We may ask how large τ should be to allow $y(t, x_0)$ to grow from value 0, 1 for $t = 0$ to value 0, 9 for $t = \tau$. The problem

$$y_t = \frac{1}{2}(y - y^3), \quad y(0, x_0) = 0, 1,$$

is solvable by

$$y(t, x_0) = \left(\frac{0,01e^t}{0,99 + 0,01e^t} \right)^{1/2},$$

hence $\tau \cong 6,045$. The behaviour of y reflects the behaviour of u (for our special f). For this f we have $m = \frac{1}{2}$, $M = 3$, hence for $t = 6$, as a

consequence of (17):

$$(18) \quad |u(6, x_0) - y(6, x_0)| \leq \varepsilon^2 \left\| |\Delta u_0| + \sum_i u_{0x_i}^2 6(e^3 - 1) \right\|_{L^\infty(\Omega)} 6e^3$$

($6e^3 \cong 120, 5$). For $\varepsilon \leq 10^{-3}$ it is reasonable to have the right side in (18) of the order 0,1 (this is, of course, a requirement on u_0). Hence, for such ε and u_0 we have $u(6, x_0) \geq y(6, x_0) - 0,1 \geq 0,8$, so that $u(t, x_0)$ growth from value 0,1 for $t = 0$ to (at least) value 0,8 for $t = 6$. Similar estimates will be made for the points x_1 at which u_0 is negative. We thus see how, in a relatively short time ($\tau = 6$), the initial variation $u_0(x_0) - u_0(x_1) = 0,2$ has been increased to the value $u(6, x_0) - u(6, x_1) \geq 1,6$. This is just a phenomenon of creation of layer structure for the solution u . The graph of $u(t, \cdot)$ consists of parts either close to +1 or close to -1 and a number of sharply sloping connections between these different parts.

We proceed now to the proof of a version of the Maximum Principle used in Theorem 1. The presented proof is a simplification and generalization of that in [6], p. 500.

Consider the equation

$$(19) \quad v_t = \sum_{i,j} (a_{ij}(t, x) v_{x_j})_{x_i} + a(t, x) v + f(t, x)$$

$x \in \Omega$, $t \in (0, T]$ ($T \leq +\infty$), with continuous coefficients fulfilling the requirements; $a(t, x) \leq \beta$ ($\beta \geq 0$ is a constant), f bounded in $[0, T] \times \bar{\Omega}$ and the main part in (19) weakly elliptic

$$(20) \quad \forall_{(t,x)} \forall_{\xi \in \mathbb{R}^n} \sum_{i,j} a_{ij}(t, x) \xi_i \xi_j \geq 0.$$

Let v satisfy the Dirichlet or Neumann condition

$$Bv = 0 \quad \text{for } x \in \partial\Omega, \quad v(0, x) = v_0(x) \quad \text{for } x \in \Omega.$$

We may then formulate

PROPOSITION 1. *The following estimate for the solution v of (19) holds*

$$(21) \quad \|v(t, \cdot)\|_{L^\infty(\Omega)} \leq (\|v_0\|_{L^\infty(\Omega)} + \sup_{0 \leq s \leq t} \|f(s, \cdot) e^{-\beta s}\|_{L^\infty(\Omega)}) e^{\beta t}.$$

Proof. If $\beta > 0$, then we may multiply (19) by $e^{-\beta t}$ to get the equation for the function $w(t, x) = v(t, x) e^{-\beta t}$

$$(22) \quad w_t = \sum_{i,j} (a_{ij}(t, x) w_{x_j})_{x_i} + (a(t, x) - \beta) w + f(t, x) e^{-\beta t}.$$

Multiplying (22) by w^{2k-1} ($k = 1, 2, 3, \dots$), integrating over Ω and by parts in the first component, we get

$$\begin{aligned} \frac{1}{2k} \frac{d}{dt} \int_{\Omega} w^{2k} dx &= -(2k-1) \int_{\Omega} w^{2k-2} \sum_{i,j} a_{ij}(t, x) w_{x_i} w_{x_j} dx \\ &\quad + \int_{\Omega} (a(t, x) - \beta) w^{2k} dx + \int_{\Omega} f(t, x) e^{-\beta t} w^{2k-1} dx. \end{aligned}$$

Omitting non-positive components and using the Hölder inequality ($p = 2k$, $q = \frac{2k}{2k-1}$) we find that

$$\frac{d}{dt} \int_{\Omega} w^{2k}(t, x) dx \leq 2k \|f(t, \cdot) e^{-\beta t}\|_{L^{\infty}(\Omega)} |\Omega|^{1/2k} \left(\int_{\Omega} w^{2k}(t, x) dx \right)^{1-1/2k}.$$

Solving this differential inequality we obtain

$$(23) \quad \|w(t, \cdot)\|_{L^{2k}(\Omega)} \leq \|v_0\|_{L^{2k}(\Omega)} + M(t) |\Omega|^{1/2k} t,$$

where $M(t) = \sup_{0 \leq s \leq t} \|f(s, \cdot) e^{-\beta s}\|_{L^{\infty}(\Omega)}$. Passing in (23) with k to $+\infty$ and returning to the proper function v we get (21). The proof is finished.

Remark 3. It is possible to get uniform in ε estimates for various norms of the solution u of (1). We will propose an example of such estimate.

Assuming convexity of Ω (for the significance of this assumption compare [2], p. 65) and the Neumann condition for u , then using the inequality

$$(24) \quad \sum_{i,j} u_{x_i x_j} u_{x_j} \cos(n, x_i) \leq 0 \quad \text{for } x \in \partial\Omega$$

proved in [2], p. 59, it is simple to reach the following estimate

$$(25) \quad \int_{\Omega} \left(\sum_i u_{x_i}^2(t, x) \right)^k dx \leq \int_{\Omega} \left(\sum_i u_{0x_i}^2 \right)^k dx e^{2kmt}$$

$k = 1, 2, 3, \dots$ (cf. [5], Lemma 2.5 for similar proof).

Acknowledgement. The author is very indebted to the Referee for a number of comments and remarks improving the manuscript of the paper.

References

- [1] N. D. Alikakos, P. W. Bates and G. Fusco, *Slow motion manifolds for a class of singular perturbation problems: the linearized equation*, J. Differential Equations 90 (1991), 81–135.
- [2] N. D. Alikakos and R. Rostamian, *Gradient estimates for degenerate diffusion equations. I*, Math. Ann. 259 (1982), 53–70.

- [3] J. Carr and R. L. Pego, *Metastable patterns in solutions of $u_t = \epsilon^2 u_{xx} - f(u)$* , Comm. Pure Appl. Math. 42 (1989), 523–576.
- [4] E. Conway, D. Hoff and J. Smoller, *Large time behavior of solutions of systems of nonlinear reaction-diffusion equations*, SIAM J. Appl. Math. 35, 1 (1978), 1–16.
- [5] T. Dłotko, *Parabolic Equations in Divergence Form. Theory of Global Solutions.*, Uniwersytet Śląski, Katowice, 1987.
- [6] T. Dłotko, *Remarks on global bounds of solutions of parabolic equations in divergence form*, Rocky Mountain J. Math. 17, 3 (1987), 499–510.
- [7] M. Freidlin, *Limit theorems for large deviations and reaction-diffusion equations*, Ann. Probability 13, 3 (1985), 639–675.
- [8] G. Fusco, *A geometric approach to the dynamics of $u_t = \epsilon^2 u_{xx} + f(u)$ for small ϵ* , Proceedings of the Stuttgart Conference in honour of J. K. Hale, K. Kirchgässner (Ed.).
- [9] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.
- [10] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics 840, Springer-Verlag, Berlin, 1981.
- [11] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Nauka, Moscow, 1967.
- [12] H. F. Weinberger, *On metastable patterns in parabolic systems*, IMA Preprint Series #164, University of Minnesota, July 1985.

INSTITUTE OF MATHEMATICS, SILESIA UNIVERSITY
Bankowa 14
40-007 KATOWICE, POLAND

Received February 18, 1991.